

REAL EIGENVALUES AND EIGENVECTORS OF 3X3 MATRICES,

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In these examples, the eigenvalues of matrices will turn out to be real values. In other words, the eigenvalues and eigenvectors are in \mathbb{R}^n .

Suppose, we have to the following matrix:

$$\mathbf{A} = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}.$$

The objective is to find the eigenvalues and the corresponding eigenvectors.

Step 1. Characteristic equation

Firstly, formulate the characteristic equation and solve it. The solutions are the eigenvalues of matrix \mathbf{A} .

If \mathbf{I} is the identity matrix of \mathbf{A} and λ is the unknown eigenvalue (represent the unknown eigenvalues), then the characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Written in matrix form, we get

$$\begin{vmatrix} 5 - \lambda & 2 & 0 \\ 2 & 5 - \lambda & 0 \\ -3 & 4 & 6 - \lambda \end{vmatrix} = 0. \quad (1)$$

Choose the row or column which are easiest to use to find the determinant of the matrix in equation (1). In other words, in this case, we will go down the last column as it contains the most zeros. This will decrease the length of the characteristic equation considerably:

$$\begin{aligned} & 0 \begin{vmatrix} 2 & 5 - \lambda \\ -3 & 4 \end{vmatrix} \\ & - 0 \begin{vmatrix} 5 - \lambda & 2 \\ -3 & 4 \end{vmatrix} \\ & + (6 - \lambda) \begin{vmatrix} 5 - \lambda & 2 \\ 2 & 5 - \lambda \end{vmatrix} = 0, \end{aligned}$$

and so

$$(6 - \lambda) \begin{vmatrix} 5 - \lambda & 2 \\ 2 & 5 - \lambda \end{vmatrix} = 0. \quad (2)$$

Simplifying further, gives

$$\begin{aligned} (6 - \lambda)[(5 - \lambda)(5 - \lambda) - 2^2] &= 0, \\ (6 - \lambda)[25 - 5\lambda - 5\lambda + \lambda^2 - 4] &= 0, \\ (6 - \lambda)[\lambda^2 - 10\lambda + 21] &= 0, \end{aligned}$$

enabling us to write a manageable form of the characteristic equation:

$$(6 - \lambda)[(\lambda - 3)(\lambda - 7)] = 0, \quad (3)$$

of which the solutions of λ are now apparent immediately:

$$\therefore \lambda = 6 \vee \lambda = 3 \vee \lambda = 7. \quad (4)$$

Lastly, there are two ways to verify if the found values are correct. For instance, the sum of these values have to be equal to the trace of \mathbf{A} , which is the sum of the main diagonal of \mathbf{A} :

$$\text{tr } \mathbf{A} = 5 + 5 + 6 = 16.$$

And, indeed, the sum of the values of λ is equal to 16 as well. We can also verify whether the product of the values of λ is equal to $\det \mathbf{A}$:

$$\begin{aligned} \det \mathbf{A} &= 0 \begin{vmatrix} 2 & 5 \\ -3 & 4 \end{vmatrix} \\ &\quad - 0 \begin{vmatrix} 5 & 2 \\ -3 & 4 \end{vmatrix} \\ &\quad + 6 \begin{vmatrix} 5 & 2 \\ 2 & 5 \end{vmatrix} \\ &= 6 \begin{vmatrix} 5 & 2 \\ 2 & 5 \end{vmatrix} \\ &= 6(5^2 - 2^2) \\ &= 126. \end{aligned}$$

And, indeed, the product of the values of λ is equal to $6 \cdot 3 \cdot 7 = 126$.

Step 2. Specify the eigenvalues

The eigenvalues of matrix \mathbf{A} are thus $\lambda = 6$, $\lambda = 3$, and $\lambda = 7$.

Step 3. Eigenvector equations

We rewrite the characteristic equation in matrix form to a system of three linear equations. As it is intended to find one or more eigenvectors \mathbf{v} , let

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (5)$$

and

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}. \quad (6)$$

In which case, we can write

$$\begin{pmatrix} 5 - \lambda & 2 & 0 \\ 2 & 5 - \lambda & 0 \\ -3 & 4 & 6 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}, \quad (7)$$

which we can then write as a system of linear equations:

$$\begin{cases} (5 - \lambda)x_1 + 2x_2 + 0x_3 = 0, \\ 2x_1 + (5 - \lambda)x_2 + 0x_3 = 0, \\ -3x_1 + 4x_2 + (6 - \lambda)x_3 = 0. \end{cases}$$

Simplifying this further, we have obtained the following *eigenvector equations*:

$$\begin{cases} (5 - \lambda)x_1 + 2x_2 = 0, \\ 2x_1 + (5 - \lambda)x_2 = 0, \\ 3x_1 - 4x_2 - (6 - \lambda)x_3 = 0. \end{cases} \quad (8)$$

Step 4. Substitute every obtained eigenvalue λ into the eigenvector equations

Eigenvalue $\lambda = 3$

Let's start with eigenvalue $\lambda = 3$. Substituting this into the eigenvector equations (8), we get

$$\begin{aligned} (5 - 3)x_1 + 2x_2 &= 0, \\ 2x_1 + (5 - 3)x_2 &= 0, \\ 3x_1 - 4x_2 - (6 - 3)x_3 &= 0. \end{aligned}$$

We can simplify this to

$$\begin{aligned} 2x_1 + 2x_2 &= 0, \\ 2x_1 + 2x_2 &= 0, \\ 3x_1 - 4x_2 - 3x_3 &= 0. \end{aligned}$$

The first equation and second equation reduce to $x_1 = -x_2$. Let's substitute x_1 in the third equation.

$$\begin{aligned} 3(-x_2) - 4x_2 - 3x_3 &= 0 \\ \therefore 7x_2 &= -3x_3. \end{aligned}$$

In other words, if $x_2 = -3$, then $x_3 = 7$, and $x_1 = 3$. And so, we can now fill in the values of \mathbf{v} in (5):

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix}. \quad (9)$$

In other words, an eigenvector with eigenvalue $\lambda = 3$ is $(3 \ -3 \ 7)^T$. (Note: we deliberately write the words 'an eigenvector', as, for instance, the eigenvector $(54 \ -54 \ 126)^T$ is an eigenvector with this eigenvalue too. As long as $x_1 = -x_2$, and $7x_2 = -3x_3$, in other words, as long as the ratios between x_1 , x_2 , and x_3 stay constant, it is *an* eigenvector of this eigenvalue. However, by convention we write the lowest possible integer values.)

We can check the validity of the eigenvector by calculating the inner product of \mathbf{A} with the eigenvector. If all went well, the outcome will be equal to the inner product of the eigenvalue with the eigenvector, in other words, $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. If we write this down in matrix notation (while, for clarity, simultaneously specifying which part is which variable), indeed, we get

$$\begin{aligned} &\underbrace{\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix}}_{\mathbf{v}} \\ &= \begin{pmatrix} 5 \cdot 3 + 2 \cdot -3 + 0 \cdot 7 \\ 2 \cdot 3 + 5 \cdot -3 + 0 \cdot 7 \\ -3 \cdot 3 + 4 \cdot -3 + 6 \cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ -9 \\ 21 \end{pmatrix} = \underbrace{3}_{\lambda} \underbrace{\begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix}}_{\mathbf{v}}. \end{aligned}$$

Eigenvalue $\lambda = 6$

In the same vein, we replace $\lambda = 6$ in the eigenvector equations of (8). We then write the following three linear equations:

$$\begin{aligned} (5 - 6)x_1 + 2x_2 &= 0, \\ 2x_1 + (5 - 6)x_2 &= 0, \\ 3x_1 - 4x_2 - (6 - 6)x_3 &= 0. \end{aligned}$$

We can simplify further:

$$\begin{aligned}x_1 &= 2x_2, \\2x_1 &= x_2, \\3x_1 &= 4x_2.\end{aligned}$$

We see that $x_1 = x_2 = 0$ is the only solution to this system of simultaneous equations. As x_3 always multiplies by 0 (which is why it does not appear anymore in this system), x_3 can take any value. An eigenvector to eigenvalue $\lambda = 6$ is therefore simply $(0 \ 0 \ 1)^T$. Checking this using the inner product of matrix \mathbf{A} with $(0 \ 0 \ 1)^T$, we get, indeed, $\lambda \mathbf{v}$:

$$\begin{aligned}&\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\&= \begin{pmatrix} 5 \cdot 0 + 2 \cdot 0 + 0 \cdot 1 \\ 2 \cdot 0 + 5 \cdot 0 + 0 \cdot 1 \\ -3 \cdot 0 + 4 \cdot 0 + 6 \cdot 1 \end{pmatrix} \\&= \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

Eigenvalue $\lambda = 7$

Substituting $\lambda = 7$, yields

$$\begin{aligned}(5 - 7)x_1 + 2x_2 &= 0, \\2x_1 + (5 - 7)x_2 &= 0, \\3x_1 - 4x_2 - (6 - 7)x_3 &= 0.\end{aligned}$$

Working this further:

$$\begin{aligned}x_1 &= x_2, \\x_1 &= x_2, \\3x_1 - 4x_2 + x_3 &= 0.\end{aligned}$$

Substituting $x_1 = x_2$ into the third equation, we get

$$\begin{aligned}3x_2 - 4x_2 &= -x_3, \\x_2 &= x_3.\end{aligned}$$

And so, we conclude

$$x_1 = x_2 = x_3. \tag{10}$$

An eigenvector to the eigenvalue $\lambda = 7$ is, thus, $(1 \ 1 \ 1)^T$. Obviously, we can check this too:

$$\begin{aligned} & \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 \cdot 1 + 2 \cdot 1 + 0 \cdot 1 \\ 2 \cdot 1 + 5 \cdot 1 + 0 \cdot 1 \\ -3 \cdot 1 + 4 \cdot 1 + 6 \cdot 1 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{aligned}$$

so, it is correct.

R codes for computing eigenvalues and eigenvectors:

```
> row1<-c(5,2,0)
> row2<-c(2,5,0)
> row3<-c(-3,4,6)
> A<-rbind(row1,row2,row3)
> A
  [,1] [,2] [,3]
row1  5  2  0
row2  2  5  0
row3 -3  4  6
> eigen(A)
eigen() decomposition
$values
[1] 7 6 3

$vectors
  [,1] [,2] [,3]
[1,] 0.5773503  0 0.3665083
[2,] 0.5773503  0 -0.3665083
[3,] 0.5773503  1 0.8551861
```